Testing Quantum Computer Controllability via Dimensional Expressivity

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Quantum Hardware and Controls

We consider quantum system linearly coupled to external controls with Hamiltonian

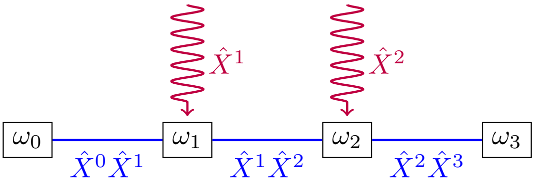
$$\hat{H}(t) = \hat{H}(t; u_1, ...u_m) = \hat{H}_0 + \sum_{j=1}^{m} u_j(t)\hat{H}_j$$

where

- $\hat{H}_0$: time-independent drift Hamiltonian (undisturbed dynamics)
- $\hat{H}_j$ $(j \geq 1)$: control Hamiltonians
- u_j: control strengths
- simplicity: $u_j(t)$ rectangular pulses with $\|u_j H_j\| \gg \|H_0\|$

Example

4 qubit system with external $\hat{\sigma}_x$ controls on qubits 1 and 2:



drift: $\hat{H}_0 = \sum_{j=0}^{3} -\frac{\omega_j}{2}\hat{Z}^j + \sum_{k=0}^{2} J_{k,k+1}\hat{X}^k\hat{X}^{k+1}$

controls: $\hat{H}_1 = \hat{X}^1$ and $\hat{H}_2 = \hat{X}^2$

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Basic VQS setup:

- initialize system in state $|\psi_0\rangle$
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- set of reachable states from $|\psi_0\rangle$ coincides with the entire state space of the quantum device $\partial B_{\mathcal{H}}/U(1)$ (unit sphere of the device Hilbert space $\mathcal{H}$ up to factors of $e^{i\alpha}$ with $\alpha \in \mathbb{R}$)

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- Note: since controls are unitary *PSC* means any initial state of the quantum device can be controlled into any other possible state of the device

Controllability for Universal Quantum Computing

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Operator Controllability (OC)

- For any target unitary $\hat{U}_t \in SU(\mathcal{H})$ there exist a time $T > 0$, a phase $\alpha \in \mathbb{R}$, and controls $u_1, \ldots, u_m$ such that the controlled evolution $\hat{U}(T; u_1, \ldots, u_m)$ satisfies

$$\hat{U}_t = e^{i\alpha}\hat{U}(T; u_1, \ldots, u_m)$$

PSC vs OC

- Pure-State Controllability: any **single state** can be controlled into any other **single state**
- Operator Controllability: any **orthonormal basis** can be controlled into any other **orthonormal basis**

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- clearly: OC $\Rightarrow$ PSC
- there exist systems that are
  - OC
  - PSC but not OC
  - not PSC
- OC $\succ$ PSC

DEA setup

A *Parametric Quantum Circuit* C (for us) is the map

$$C: \text{ parameter space } \mathcal{P} \to \text{quantum device state space } \mathcal{S};\ \vartheta \mapsto |\psi(\vartheta)\rangle\,,$$

i.e., C contains both the gate sequence and the initial state $|\psi_0\rangle$.

"Optimal" Circuit

- *maximally expressive:* be able to generate all (physically relevant) states
- *minimal:* not contain "unnecessary" parameters/gates

Finding redundant parameters

ϑ_k is redundant iff $\partial_k C(\vartheta)$ is a linear combination of the $\partial_j C(\vartheta)$ with $j \neq k$

⇒ inductively check each real partial Jacobian J_k of C

$$J_k = \begin{pmatrix} | & & | \\ \Re\partial_1 C & \cdots & \Re\partial_k C \\ | & & | \\ & & \\ | & & | \\ \Im\partial_1 C & \cdots & \Im\partial_k C \\ | & & | \end{pmatrix}$$

for invertibility (e.g., by computing the smallest eigenvalue of $S_k := J_k^* J_k$)

⇒ Assuming $\vartheta_1, \ldots, \vartheta_{k-1}$ are independent, then ϑ_k is dependent if and only if $\det S_k = 0$. Note $S_k \geq 0$, so we can check $\lambda_{\min} > \varepsilon$ to conclude $\det S_k \neq 0$.

Example: $C(\vartheta) = \hat{R}_Z(\vartheta_2)\hat{R}_X(\vartheta_1)\,|0\rangle$

$$C(\vartheta) = \hat{R}_Z(\vartheta_2)\hat{R}_X(\vartheta_1)\,|0\rangle = \begin{pmatrix} \cos\frac{\vartheta_1}{2}\cos\frac{\vartheta_2}{2} - i\cos\frac{\vartheta_1}{2}\sin\frac{\vartheta_2}{2} \\ -i\sin\frac{\vartheta_1}{2}\cos\frac{\vartheta_2}{2} + \sin\frac{\vartheta_1}{2}\sin\frac{\vartheta_2}{2} \end{pmatrix} \begin{pmatrix} |0\rangle \\ |1\rangle \end{pmatrix}$$

yields

$$J_1 = \frac{1}{2}\begin{pmatrix} -\sin\frac{\vartheta_1}{2}\cos\frac{\vartheta_2}{2} \\ \cos\frac{\vartheta_1}{2}\sin\frac{\vartheta_2}{2} \\ \sin\frac{\vartheta_1}{2}\sin\frac{\vartheta_2}{2} \\ -\cos\frac{\vartheta_1}{2}\cos\frac{\vartheta_2}{2} \end{pmatrix} \quad \text{and} \quad J_2 = \frac{1}{2}\begin{pmatrix} -\sin\frac{\vartheta_1}{2}\cos\frac{\vartheta_2}{2} & -\cos\frac{\vartheta_1}{2}\sin\frac{\vartheta_2}{2} \\ \cos\frac{\vartheta_1}{2}\sin\frac{\vartheta_2}{2} & \sin\frac{\vartheta_1}{2}\cos\frac{\vartheta_2}{2} \\ \sin\frac{\vartheta_1}{2}\sin\frac{\vartheta_2}{2} & -\cos\frac{\vartheta_1}{2}\cos\frac{\vartheta_2}{2} \\ -\cos\frac{\vartheta_1}{2}\cos\frac{\vartheta_2}{2} & \sin\frac{\vartheta_1}{2}\sin\frac{\vartheta_2}{2} \end{pmatrix}.$$

Hence

$$S_1 = J_1^* J_1 = \frac{1}{4} \quad \text{and} \quad S_2 = J_2^* J_2 = \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{4} \end{pmatrix}$$

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yields

$$J_2 = \frac{1}{2}\begin{pmatrix} -\sin\frac{\vartheta_1}{2}\cos\frac{\vartheta_2}{2} - \cos\frac{\vartheta_1}{2}\sin\frac{\vartheta_2}{2} & -\cos\frac{\vartheta_1}{2}\sin\frac{\vartheta_2}{2} - \sin\frac{\vartheta_1}{2}\cos\frac{\vartheta_2}{2} \\ 0 & 0 \\ 0 & 0 \\ -\cos\frac{\vartheta_1}{2}\cos\frac{\vartheta_2}{2} + \sin\frac{\vartheta_1}{2}\sin\frac{\vartheta_2}{2} & \sin\frac{\vartheta_1}{2}\sin\frac{\vartheta_2}{2} - \cos\frac{\vartheta_1}{2}\cos\frac{\vartheta_2}{2} \end{pmatrix}.$$

Hence

$$S_1 = J_1^* J_1 = \frac{1}{4} \quad \text{and} \quad S_2 = J_2^* J_2 = \frac{1}{4}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{with} \quad \sigma(S_2) = \left\{0, \frac{1}{2}\right\}.$$

Hybrid quantum-classical implementation of DEA

- Use quantum device to measure matrices $S_k = J_k^* J_k = \begin{pmatrix} S_{k-1} & A_k \\ A_k^* & c_k \end{pmatrix}$.
  - requires 1 ancilla qubit

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$\Rightarrow$ Memory: $O(N^2)$
CPU calls: $O(N^4)$
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$\Rightarrow$ Polynomial in #parameters N

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  $\Rightarrow$ image manifold = set of reachable state = $\partial B_{\mathcal{H}}/U(1)$ if and only if the number of independent parameters in the quantum circuit = $\dim(\partial B_{\mathcal{H}}/U(1))$
- find control based quantum circuit and check dimensional expressivity

PSC Circuit

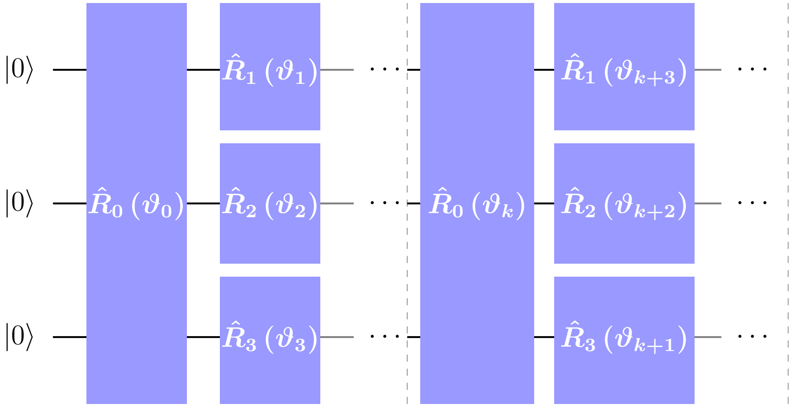
Idea: Trotter product n layers of rotations around drift and control hamiltonians

$$C(\vartheta) = L(\vartheta_{(n-1)(m+1)}, \ldots, \vartheta_{(n-1)(m+1)+m}) \cdots L(\vartheta_0, \ldots, \vartheta_m) \,|\psi_0\rangle$$

with the k^{th} layer

$$L(\vartheta_{k(m+1)}, \ldots, \vartheta_{k(m+1)+m}) = \underbrace{e^{-i\frac{\vartheta_{k(m+1)+m}}{2}\hat{H}_m}}_{\hat{R}_m(\vartheta_{k(m+1)+m})} \cdots \underbrace{e^{-i\frac{\vartheta_{k(m+1)+1}}{2}\hat{H}_1}}_{\hat{R}_1(\vartheta_{k(m+1)+1})} \underbrace{e^{-i\frac{\vartheta_{k(m+1)}}{2}\hat{H}_0}}_{\hat{R}_0(\vartheta_{k(m+1)})}$$

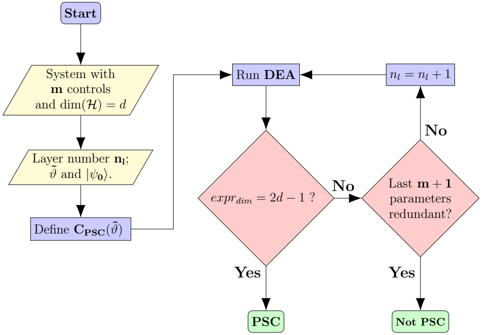
PSC Circuit Example



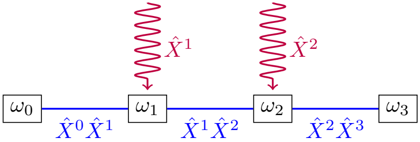
PSC Analysis

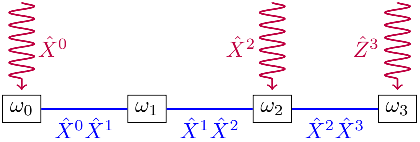
DEA with randomly chosen parameters terminates if

- maximal expressivity is reached ⇒ PSC
- full redundant layer is reached ⇒ not PSC



PSC Examples (need 31 independent parameters)





- coupling strengths (MHz): $J_{0,1} = 170,\ J_{1,2} = 220,\ J_{2,3} = 150$
- qubit frequencies (GHz): $\omega_0 = 5.40$, $\omega_1 = 5.30$, $\omega_2 = 5.42$, $\omega_3 = 5.37$

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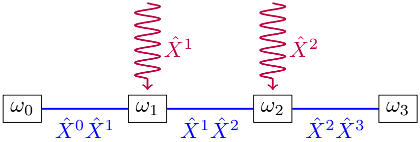

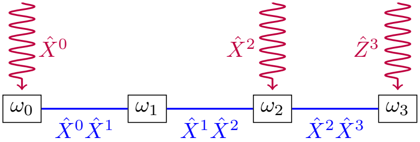

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- first 31 parameters are all independent $\Rightarrow$ PSC

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- only 29 independent parameters before redundant layer $\Rightarrow$ not PSC

OC as PSC problem

- need to control arbitrary $\hat{U} \in U(\mathcal{H})/U(1) \hookrightarrow L(\mathcal{H}) \cong \mathcal{H} \otimes \mathcal{H}$
- $\mathcal{H} \otimes \mathcal{H}$ can be represented as qubit system of double size
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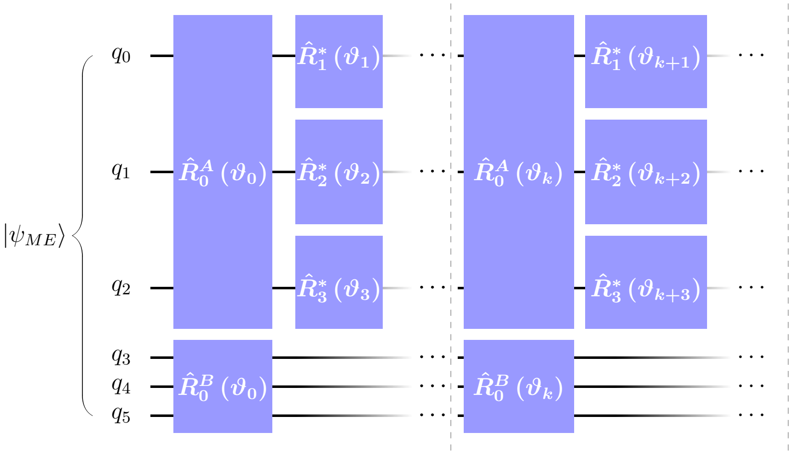
- now have A (original) and B (copy) subsystems
- prepare maximally entangled state $|\psi_{ME}\rangle = \sum_{i=0}^{d-1} \frac{1}{\sqrt{d}} |e_i\rangle \otimes |e_i\rangle$
- consider PSC circuit in A and drift only in B
- do DEA

OC Circuit Example



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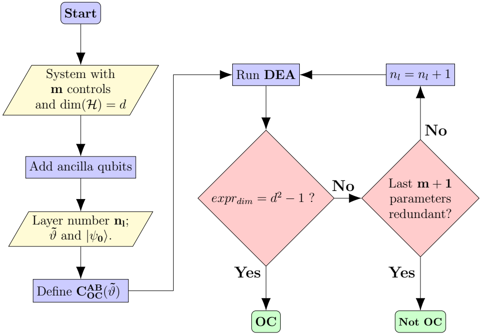
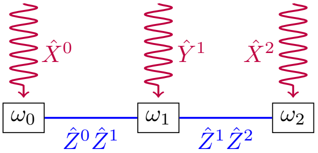
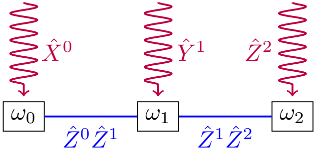


OC Examples (need 63 independent parameters)





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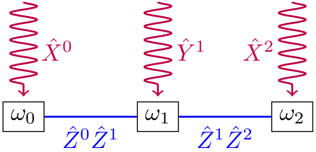

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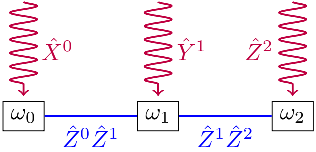
- 63 of first 64 parameters are independent $\Rightarrow$ OC



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What have we got? arXiv:2308.00606

- hybrid quantum-classical algorithms to test PSC and OC
- can run directly on quantum hardware with ancilla qubits
- resource efficient hardware design
  - if a control is always redundant, you don't need it
  - deduce minimum number of local controls given set of potential controls
- can identify whether hardware is universal or at least enough for VQS

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- study systems with non-local controls

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Future ideas?

- study systems with non-local controls
- How does removal of controls affect control time (quantum speed limit)?